Fourier Analysis 03-26

Review:

Def. A function $f: \mathbb{R} \to \mathbb{C}$ is said to be of moderate decrease if ① f is cts on $\mathbb{R}$.

② $\exists$ a constant $A > 0$ such that

$$|f(x)| \leq \frac{A}{1+x^2}, \quad \forall x \in \mathbb{R}.$$

Let $\mathcal{M}(\mathbb{R})$ be the collection of all functions on $\mathbb{R}$ of moderate decrease.

Def. Let $f \in \mathcal{M}(\mathbb{R})$. The Fourier transform of f is

$$\hat{f}(\xi) = \int_{-\infty}^{\infty} f(x)\, e^{-2\pi i \xi \cdot x}\, dx, \quad \xi \in \mathbb{R}$$

§5.2 Some properties of the Fourier transform.

Prop 1: Let $f \in \mathcal{M}(\mathbb{R})$. Then the following hold:

① $f(x+h) \xmapsto{\mathcal{F}} \hat{f}(\xi) \cdot e^{2\pi i h \xi}, \quad \forall h \in \mathbb{R}$

② $f(x)\, e^{-2\pi i h x} \xmapsto{\mathcal{F}} \hat{f}(\xi + h), \quad \forall h \in \mathbb{R}.$

③ $f(\delta x) \xmapsto{\mathcal{F}} \frac{1}{\delta} \hat{f}\left(\frac{\xi}{\delta}\right), \quad \forall \delta > 0.$

④ $f'(x) \xmapsto{\mathcal{F}} (2\pi i \xi) \hat{f}(\xi), \quad \text{if } f' \in M(\mathbb{R})$

⑤ $-2\pi i x f(x) \xmapsto{\mathcal{F}} \frac{d \hat{f}(\xi)}{d \xi}, \quad \text{if } x f(x) \in M(\mathbb{R}).$

Pf: ① $\displaystyle\int_{-\infty}^{\infty} f(x+h) e^{-2\pi i \xi x} dx$

$$= \lim_{N\to\infty} \int_{-N}^{N} f(x+h) e^{-2\pi i \xi x} dx$$

$$= \lim_{N\to\infty} \int_{-N+h}^{N+h} f(y) e^{-2\pi i \xi (y-h)} dy$$

$$= e^{2\pi i \xi y} \lim_{N\to\infty} \int_{-N+h}^{N+h} f(y) e^{-2\pi i \xi y} dy$$

$$= e^{2\pi i \xi y} \hat{f}(\xi).$$

Similarly we can prove ② and ③.

④ $$\int_{-\infty}^{\infty} f'(x)\, e^{-2\pi i \xi x}\, dx$$

$$= \lim_{N\to\infty} \int_{-N}^{N} f'(x)\, e^{-2\pi i \xi x}\, dx$$

$$= \lim_{N\to\infty} \left[f(x)\, e^{-2\pi i \xi x} \Big|_{-N}^{N} - \int_{-N}^{N} f(x)\cdot(-2\pi i \xi)\, e^{-2\pi i \xi x}\, dx \right]$$

$$= 2\pi i \xi\, \hat{f}(\xi).$$

⑤. $$\frac{\hat{f}(\xi+\Delta\xi) - \hat{f}(\xi)}{\Delta\xi} = \int_{-\infty}^{\infty} f(x)\cdot \frac{e^{-2\pi i(\xi+\Delta\xi)x} - e^{-2\pi i \xi x}}{\Delta\xi}\, dx$$

$$= \int_{-\infty}^{\infty} f(x)\cdot e^{-2\pi i \xi x}\cdot \frac{e^{-2\pi i \Delta\xi x} - 1}{\Delta\xi}\, dx$$

It is clear that

$$\lim_{\Delta\xi\to 0} \frac{e^{-2\pi i \Delta\xi x} - 1}{\Delta\xi} = -2\pi i x$$

We claim that

$$\left| \frac{e^{-2\pi i \Delta\xi x} - 1}{\Delta\xi} \right| \leq 2\pi |x|.$$

To see it, notice that

$$e^{-2\pi i \Delta\xi x} - 1 = e^{-\pi i \Delta\xi x} \cdot \left[e^{-\pi i \Delta\xi x} - e^{\pi i \Delta\xi x} \right]$$

$$= -2i\, e^{-\pi i \Delta\xi x} \cdot \sin(\pi \Delta\xi x)$$

So
$$\frac{\left| e^{-2\pi i \Delta\xi x} - 1 \right|}{|\Delta\xi|} = \frac{2\left|\sin(\pi\Delta\xi x)\right|}{|\Delta\xi|}$$

$$\leq 2\pi|x|.$$

(Lebesgue dominated convergence Thm)

Let (f_t) be a family of functions in $M(\mathbb{R})$ such that

① $\sup_t |f_t| \leq F$ with $\int_{-\infty}^{\infty} F(x)dx < \infty$

② $\lim_{t\to t_0} f_t(x) = g(x)$, $\forall\ x \in \mathbb{R}$ for some $g \in M(\mathbb{R})$.

Then
$$\lim_{t\to t_0} \int_{\mathbb{R}} f_t(x)\,dx = \int_{\mathbb{R}} g(x)\,dx.$$

Now applying DCT to

$$f(x) \cdot e^{-2\pi i \xi x} \cdot \frac{e^{-2\pi i \Delta\xi x} - 1}{\Delta\xi}$$

we obtain the desired result.

Thm (Riemann-Lebesgue Lemma)

Let $f \in M(\mathbb{R})$. Then

$$\lim_{\xi \to \infty} \hat{f}(\xi) = 0.$$

pf. Notice that

$$\hat{f}(\xi) = \int_{-\infty}^{\infty} f(x) e^{-2\pi i \xi x} dx$$

$$= \int_{-\infty}^{\infty} f(x+\tfrac{1}{2\xi}) e^{-2\pi i \xi (x+\frac{1}{2\xi})} dx$$

$$= \int_{-\infty}^{\infty} -f(x+\tfrac{1}{2\xi}) e^{-2\pi i \xi x} dx$$

Hence

$$\hat{f}(\xi) = \int_{-\infty}^{\infty} \frac{f(x) - f(x+\frac{1}{2\xi})}{2} e^{-2\pi i \xi x} dx$$

Hence

$$|\hat{f}(\xi)| \leq \frac{1}{2} \int_{-\infty}^{\infty} \left| f(x) - f(x+\tfrac{1}{2\xi}) \right| dx$$

$$\to 0 \quad \text{as} \quad \xi \to \infty. \qquad \blacksquare$$

Examples: (1) Calculate $\hat{f}$ for $f(x) = e^{-\pi x^2}$.

$$\hat{f}(\xi) = \int_{\mathbb{R}} e^{-\pi x^2} e^{-2\pi i \xi x} dx$$

$$\frac{d\hat{f}(\xi)}{d\xi} = \int_{\mathbb{R}} (-2\pi i x)\, e^{-\pi x^2} e^{-2\pi i \xi x} dx$$

$$= \int_{\mathbb{R}} i \left(e^{-\pi x^2}\right)' e^{-2\pi i \xi x} dx$$

$$= i\, e^{-\pi x^2} \Big|_{-\infty}^{\infty} - \int_{\mathbb{R}} i\, e^{-\pi x^2} \cdot (-2\pi i \xi)\, e^{-2\pi i \xi x} dx$$

$$= -2\pi \xi \cdot \int_{\mathbb{R}} e^{-\pi x^2} e^{-2\pi i \xi x} dx$$

$$= -2\pi \xi \cdot \hat{f}(\xi).$$

$$e^{\pi \xi^2} \cdot \left(\hat{f}'(\xi) + 2\pi \xi \hat{f}(\xi) \right) = 0$$

i.e. $$\frac{d\left(e^{\pi \xi^2} \hat{f}(\xi)\right)}{d\xi} = 0$$

$$\Rightarrow \quad e^{\pi \xi^2} \hat{f}(\xi) = \text{const.}$$

$$\Rightarrow \quad \hat{f}(\xi) = c \cdot e^{-\pi \xi^2}.$$

Finally $\hat{f}(0) = \int_{-\infty}^{\infty} e^{-\pi x^2} dx = 1.$

So $C = 1.$

Example 2. Calculate $\hat{f}$ for $f(x) = e^{-|x|}.$

$$\hat{f}(\xi) = \int_{-\infty}^{\infty} e^{-|x|} e^{-2\pi i \xi x} dx$$

$$= \int_{0}^{\infty} e^{-x} e^{-2\pi i \xi x} dx + \int_{-\infty}^{0} e^{x} e^{-2\pi i \xi x} dx$$

$$= \int_{0}^{\infty} e^{x(-1-2\pi i \xi)} dx + \int_{-\infty}^{0} e^{x(1-2\pi i \xi)} dx$$

$$= \left. \frac{e^{x(-1-2\pi i \xi)}}{-1-2\pi i \xi} \right|_{0}^{\infty} + \left. \frac{e^{x(1-2\pi i \xi)}}{1-2\pi i \xi} \right|_{-\infty}^{0}$$

$$= \frac{1}{1+2\pi i \xi} + \frac{1}{1-2\pi i \xi}$$

$$= \frac{2}{1+4\pi^2 \xi^2}.$$

Ex. Let $a>0$. Calculate the Fourier transform of e^{-ax^2} and $e^{-a|x|}$.

- Notice that letting $f(x) = e^{-\pi x^2}$, then

$$g(x) = e^{-ax^2} = f\left(\sqrt{\tfrac{a}{\pi}}\,x\right)$$

Hence

$$\hat{g}(\xi) = \sqrt{\frac{\pi}{a}} \cdot \hat{f}\left(\xi \cdot \sqrt{\frac{\pi}{a}}\right)$$
$$= \sqrt{\frac{\pi}{a}} \cdot e^{-\pi \cdot \left(\xi^2 \cdot \frac{\pi}{a}\right)}$$
$$= \sqrt{\frac{\pi}{a}} \cdot e^{-\pi^2 \xi^2 / a}.$$

- $g(x) = e^{-a|x|} = f(ax)$ where $f(x) = e^{-|x|}$

$$\hat{g}(\xi) = \frac{1}{a}\hat{f}\left(\frac{\xi}{a}\right) = \frac{1}{a} \cdot \frac{2}{1 + 4\pi^2 \frac{\xi^2}{a^2}} = \frac{2a}{a^2 + 4\pi^2 \xi^2}.$$